Holevo's bound is a limit on how much classical information may be retrieved from the quantum system.

**Theorem (Holevo bound):** For any ensemble \( \{p_x, s_x\}_{x \in \mathcal{X}} \):

\[
\text{Acc}(\{p_x, s_x\}) \leq \frac{1}{\mathcal{X}} \left( \sum_{x} p_x s_x - \sum_{x} p_x S(s_x) \right)
\]

\( \mathcal{X}(\{p_x, s_x\}) \)

"Holevo x-quantity"

**Proof:** Let \((E_y)_{y \in \mathcal{Y}}\) be any POVM in \(B(H)\). Define the following state (purely mathematically):

\[
\rho_{\text{QMA}} = \sum_{x \in \mathcal{X}} p_x |x_x\rangle \langle x_x| \otimes \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} p_x |x_x\rangle \langle x_x| \otimes |y_x\rangle \langle y_x| \otimes \mathbf{0} \in \mathbf{R}(L \otimes \mathbf{X} \otimes \mathbf{Y})
\]

On states of the form \(1_{\mathbb{C}^2} \otimes 1_{\mathbb{C}^M} \otimes 1_{\mathbb{H}^4} \in \mathbf{H} \otimes \mathbf{E} \otimes \mathbf{F}\) of the QMA system, define the following map:

\[
\mathbf{U}(1_{\mathbb{C}^2} \otimes 1_{\mathbb{C}^M} \otimes 1_{\mathbb{H}^4}) = \sum_{y \in \mathcal{Y}} \sqrt{E_y} 1_{\mathbb{C}^2} \otimes 1_{\mathbb{C}^M} \otimes 1_{\mathbb{H}^4}
\]

For now, this is just a mathematical definition, but it will be useful to model the measurement interaction.
U preserves the inner product of states its action (i.e., it is an isometry), since for any $|x\rangle, |y\rangle \in H$: 

$$
\left( \sum_y \sqrt{E_y} |x\rangle \otimes |y\rangle \langle y| \right)^+ \left( \sum_y \sqrt{E_y} |x\rangle \otimes |y\rangle \langle y| \right)
$$

$$
= \left( \sum_y \sqrt{E_y} |x\rangle \otimes |y\rangle \langle y| \right)^+ \left( \sum_y \sqrt{E_y} |x\rangle \otimes |y\rangle \langle y| \right)
$$

$$
= \sum_{y, y'} \sqrt{E_y} \langle x | y \rangle \langle y' | \langle y| \cdot \sum_{y, y'} \sqrt{E_{y'}} \langle y' | x \rangle \langle x | y \rangle
$$

$$
= \sum_{y, y'} \langle x | y \rangle \langle y | x \rangle = \delta_{y, y'}
$$

Therefore, we can extend $U$ to a unitary operator on the whole system $QMA$, i.e., $U : \mathcal{H}_Q \otimes \mathcal{H}_M \otimes \mathcal{H}_A \rightarrow \mathcal{H}_Q \otimes \mathcal{H}_M \otimes \mathcal{H}_A$.

Let us take any such extension and define:

$$
\gamma_{QMA} = (A_Q \otimes U_{QMA}) \gamma_{QMA} (A_Q \otimes U_{QMA})^+
$$

$$
= \sum_x \sum_{y, y'} p_x 1_{y < x} \otimes \sqrt{E_y} p_x \sqrt{E_{y'}} \otimes |y \rangle \langle y' | y > x
$$

Now we trace out the $A$ (ancilla) system:

$$
\gamma_{QMA} = tr_A[\gamma_{QMA}] = \sum_x \sum_{y, y'} p_x 1_{y < x} \otimes \sqrt{E_y} p_x \sqrt{E_{y'}} \otimes |y \rangle \langle y' | y > x
$$

Now, the system contains the measurement result.
Next, trace out the $\mathcal{Q}$ subsystem:
\[
\mathcal{Q}_{LM} = \sum_x \sum_y p_x \left| x \right> \left< y \right|_{\mathcal{M}} \text{Tr} [\left| \left< y \right|_{\mathcal{M}} \mathcal{E}_y \mathcal{E}_y \left| x \right> \right|_{\mathcal{M}}] = \text{Tr} [\mathcal{E}_y \mathcal{E}_y \mathcal{E}_y] = \text{Tr} [\mathcal{E}_y \mathcal{E}_y] = \rho (y | x).
\]

This is basically a classical state, corresponding to the random variables $X$ and $Y$.

Now we use strong subadditivity (SSA) in the form $I(A : B) \leq I(A : BC)$:
\[
I(L : \mathcal{Q} \mathcal{M})_p = S(L)_p + S(\mathcal{Q} \mathcal{M})_p - S(L \mathcal{Q} \mathcal{M})_p
\]
\[
\overset{\text{invariance}}{=} S(L)_p + S(\mathcal{Q} \mathcal{M})_p - S(L \mathcal{Q} \mathcal{M})_p
\]
\[
\overset{\text{underminatry}}{=} I(L : \mathcal{Q} \mathcal{M})_p \geq I(L : \mathcal{Q} \mathcal{M})_p
\]
\[
\overset{\text{strong subadditivity}}{=} I(L : M)_p
\]

Now compute the left-hand-side and right-hand-side of $(\ast)$:

In the state $\rho^{\mathcal{Q} \mathcal{M}}$, the $10 \otimes 10 \otimes 10 \otimes 10$ adds only a few zero-eigenvalue, to the spectrum of the state $\rho^{\mathcal{Q} \mathcal{M}}$ (and its marginals), so that
\[
I(L : \mathcal{Q} \mathcal{M})_p = S(L)_p + S(\mathcal{Q} \mathcal{M})_p - S(L \mathcal{Q} \mathcal{M})_p
\]
\[
= S(L)_p + S(\mathcal{Q})_p - S(L \mathcal{Q})_p
\]
\[
= I(L : \mathcal{Q})_p = S(\sum_x p_x s_x) - \sum_x p_x S(p_x).
\]

Exercise 4.3
Due to $S^{\mu} = \sum_{\mathbf{x}} \sum_{\mathbf{y}} p(x, y) \log \frac{1}{p(x, y)}$, we have

$$I(L : M) \geq I(X : Y) p(x, y).$$

(Also follows from Exercise 4.3.)

Thus, the inequality (*) gives:

$$S(\sum_{x} p_x s_x) - \sum_{x} p_x S(p_x) \geq I(X : Y) p(x, y).$$

As this holds for all POVMs $(E_y)_{y \in Y}$, we have proven the claimed upper bound on the accessible information.

\[ \square \]

**Remark:** In general, Holevo's upper bound on the accessible information is not tight (see also Exercise 4.5(d)). No general method for computing $\text{Acc}(\{p_x, s_x\})$ is known, but at least Holevo's theorem gives a easy-to-compute upper bound on it.

**Remark:** Holevo's bound shows that in a quantum system $\mathcal{H}$ of dimension $d$, one cannot store more than $(\log d)$ classical bits of information since

$$I(X : Y) \leq S(\sum_{x} p_x s_x) - \sum_{x} p_x S(p_x) \leq S(\sum_{x} p_x s_x) \leq \log d.$$

Thus, if $\mathcal{H} = (\mathcal{H}^n)^{\otimes m} = \mathcal{H}^{(2^n)}$ is a system composed of $m$ qubits, we cannot store more than $\log 2^n = n$ classical bits of information in it. Holevo's bound shows that the use of non-orthogonal states $p_x$ or the use of coherence (non-commuting states) does not help.

(In fact: $I(X : Y) \leq \chi < H(X)$ if two $p_x, s_x$ (with $p_x \neq 0, s_x \neq 0$) are not orthogonal (see Nielsen and Chuang, Theorem 11.10).)
Consequence of Holevo's bound (more details in Chapter II.3):

If $T: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is a "quantum channel" over which we want to transmit classical information by encoding into some "individual" channel inputs $\rho_x$ and decode with measurement $E=(E_y)$ to give a "classical channel" $X \rightarrow Y$, the following is an upper bound on the capacity:

$$C_{1,\infty}(T) \leq \max \{ \text{dec}(\{\rho_x, T(\rho_x)\}) \} \leq \max \left( S(\Sigma \rho_x T(\rho_x)) - \sum_x \rho_x S(T(\rho_x)) \right)$$

"individual" separable input states

upper bound on mutual information

Holevo bound

maximize overall input ensemble $\{\rho_x, \rho_x\}$