Exercise set 5

Due date: Wednesday, June 29, before class

Note: Starred* problems give bonus points.

Exercise 5.1 (Shannon vs. von Neumann entropy):
For an ensemble $E = \{p_i, |\psi_i\rangle\}_{i=1}^I$ of pure states $|\psi_i\rangle \in \mathbb{C}^d$, define $\rho := \sum_i p_i |\psi_i\rangle \langle \psi_i|$. The goal of this exercise is to show that

$$S(\rho) \leq H(\{p_i\}),$$

i.e. that the von Neumann entropy of the averaged ensemble state never exceeds the Shannon entropy of the ensemble probability distribution $\{p_i\}$.

Hint: The following proof strategy is similar to our proof of Holevo’s bound.

(a) Define $\rho_{ABC} := \sum_i p_i |i\rangle_A \langle i| \otimes |i\rangle_B \langle i| \otimes |0\rangle_C \langle 0| \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$. Compute $I(A:BC)_\rho$.

(b) Let $U_{BC} \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ be an operator satisfying $U_{BC} |i\rangle_B \otimes |0\rangle_C = |i\rangle_B \otimes |\psi_i\rangle_C$, and define $\sigma_{ABC} := (1_A \otimes U_{BC}) \rho_{ABC} (1_A \otimes U_{BC}^\dagger)$. Why can $U_{BC}$ be chosen as a unitary, and what is $I(A:BC)_{\sigma}$?

(c) Compute $I(A:C)_{\sigma}$. Comparing to $I(A:BC)_{\sigma}$, how can one conclude (1)?

Background: A variant of Schumacher’s data compression theorem says that storing a product state $|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$ from the i.i.d. ensemble $\mathcal{E}^\otimes n$ can be done (with high average fidelity) on a quantum memory of $\approx nS(\rho)$ qubits. Shannon’s data compression theorem however requires at least $\approx nH(\{p_i\})$ classical bits to store the index value $(i_1, \ldots, i_n)$. Result (1) therefore implies that directly compressing the quantum data needs less memory than storing the classical index, but the memory needs to be of a quantum kind. (It can be shown that equality in (1) holds only if the pure states $|\psi_i\rangle$ are all orthogonal; see Nielsen&Chuang, Theorem 11.10.) △

*Exercise 5.2 (The transposition map):
Let $\{|i\rangle\}_{i=1}^d$ be an orthonormal basis of a Hilbert space $\mathbb{C}^d$. The transposition w.r.t. this basis is defined to be the map

$$\theta : \mathcal{B}(\mathbb{C}^d) \mapsto \mathcal{B}(\mathbb{C}^d), \quad \theta(X) := \sum_{i,j=1}^d |j\rangle \langle i| X |j\rangle \langle i|.$$

(a) Argue in how far this agrees with the common notion of matrix transposition.

(b) Show: If $X \geq 0$, then $\theta(X) \geq 0$. Remark: This means that $\theta$ is a positive map. △

*Exercise 5.3 (Basis dependence of the transposition):
Let $\theta$ be the transposition w.r.t. the orthonormal basis $\{|i\rangle\}_{i=1}^d \subset \mathbb{C}^d$, and $\tau$ the transposition w.r.t. orthonormal basis $\{|\alpha\rangle\}_{\alpha=1}^d \subset \mathbb{C}^d$. Show that there exists a unitary $V \in \mathcal{B}(\mathbb{C}^d)$ such that:

$$\forall X \in \mathcal{B}(\mathbb{C}^d) : \quad \tau(X) = V \theta(X) V^\dagger.$$

This means that any two transpositions are simply related by a unitary conjugation. △
**Exercise 5.4 (Entanglement negativity):**

Let $|\psi_{AB}\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |i\rangle_A |i\rangle_B \in \mathbb{C}^d \otimes \mathbb{C}^d$ be a pure state, where $\{|i\rangle\}_{i=1}^{d}$ are orthonormal bases in $A$ resp. $B$, and define $\rho_{AB} := |\psi_{AB}\rangle \langle \psi_{AB}|$. Let $\theta_A$ be the transposition w.r.t. the basis $\{|i\rangle\}_A$.

(a) Compute $\rho_{AB}^\Gamma := (\theta_A \otimes \text{id}_B)(\rho_{AB})$, where $\text{id}_B(X_B) := X_B$ for all $X_B \in \mathcal{B}(\mathbb{C}^d)$ defines the identity map. What are the eigenvalues of $\rho_{AB}^\Gamma$?

*Hint:* It may be helpful to consider the case $d = 2$ explicitly first.

*Remark:* $\rho_{AB}^\Gamma$ is called the partial transposition of $\rho_{AB}$.

(b) Compute the trace norm $\|\rho_{AB}^\Gamma\|_1$ (which is known as the entanglement negativity $\mathcal{N}(\rho_{AB})$ of the state $\rho_{AB}$).

*Remark:* Any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written in a similar form as above (see Exercise 4.2). And if we take the transposition w.r.t. a different basis, this does not change the eigenvalues of $\rho_{AB}^\Gamma$ (see Exercise 5.3). Thus, the entanglement negativity of a pure state depends only on its Schmidt coefficients $\{\lambda_i\}_i$.

(c*) What is the maximum $\mathcal{N}(\rho_{AB})$ among all pure states $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$?

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**Exercise 5.5 (Steering for pure states):**

*Background:* Given a state $\rho_Q \in \mathcal{B}(\mathbb{C}^d)$, there are generally many ensembles $\{p_k, \rho_k\}$ that satisfy $\rho_Q = \sum_k p_k \rho_k$. Any such ensemble can be identified with the classical-quantum state $\rho_{CQ} := \sum_k p_k |k\rangle_C \langle k| \otimes \rho_k$. In the lecture, we mentioned that a purification $|\psi_{QR}\rangle$ of $\rho_Q$ is in some sense more useful than any such classical-quantum state, and this is why we have based our definition of compression accuracy on the fidelity of a purification. In the following, we want to show that any such $\rho_{CQ}$ can indeed be obtained from $|\psi_{QR}\rangle$ simply by performing a measurement on the reference system $R$.

Let $\rho_Q = \sum_i \lambda_i |i\rangle_Q \langle i| \in \mathbb{C}^d \otimes \mathbb{C}^d$ be an eigendecomposition and $|\psi_{QR}\rangle = \sum_i \sqrt{\lambda_i} |i\rangle_Q |i\rangle_R \in \mathbb{C}^d \otimes \mathbb{C}^d$ be a purification of $\rho_Q$. If one performs a POVM measurement $\{E_k\}$ on the $R$-system and obtains a particular outcome $k$, then the corresponding state of the $Q$-system will be $\sigma_k = \text{tr}_R((1_Q \otimes E_k)\rho_{QR})/q_k$, and this outcome will occur with probability $q_k = \text{tr}((1_Q \otimes E_k)\rho_{QR})$.

(a) Express $q_k \sigma_k$ in terms of $\sqrt{\lambda}$ and the transposition $(E_k)^T$ of $E_k$.

(b) Let now an ensemble $\{p_k, \rho_k\}$ with average state $\rho_Q = \sum_k p_k \rho_k$ be given, and let $|\psi_{QR}\rangle$ be a purification of $\rho_Q$ as above. Write down a POVM measurement $\{E_k\}$ on $R$ such that, for each outcome $k$, one obtains the state $\rho_k$ on $Q$ and that this happens with probability $p_k$.

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