Ergänzungen zur Vorlesung

Kanonische Formulierung der ART

von Domenico Giulini

Hodge-Duality

1 Exterior product and algebra

Let V be a real n-dimensional vector space, V^* its dual space and $T^pV^* = V^* \otimes \cdots \otimes V^*$ its p-fold tensor product. We will follow standard tradition to define *forms*, i.e. the antisymmetric tensor product on the dual vector space V^* rather than on V. Clearly, all constructions that are to follow could likewise be made in terms of Vrather than V^*.

T^pV^* carries a representation π_p of S_p, the symmetric group (permutation group) of p objects, given by

$$\pi_P : S_p \to \mathrm{End}(T^pV^*), \quad \pi_p(\sigma)\big(\alpha_1 \otimes \cdots \otimes \alpha_p\big) := \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)} \tag{1}$$

and linear extension to sums of tensor products. On T^pV^* we define the linear operator of antisymmetrisation by

$$\mathrm{Alt}_p := \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\, \pi_{\mathrm{p}}\,, \tag{2}$$

where $\mathrm{sign} : S_p \to \{1, -1\} \cong \mathbb{Z}_2$ is the sign-homomorphism. This linear operator is idempotent (i.e. a projection operator) and its image of T^pV^* under Alt_p is the subspace of totally antisymmetric tensor-products. We write

$$\pi_p(T^pV^*) =: \bigwedge^p V^*\,. \tag{3}$$

Clearly

$$\dim\left(\bigwedge^p V^*\right) = \begin{cases} \binom{n}{p} & \text{for } p \leq n\,, \\ 0 & \text{for } p > n\,. \end{cases} \tag{4}$$

We set

$$\bigwedge V^* := \bigoplus_{p=0}^{n} \bigwedge^p V^*\,. \tag{5}$$

Let $\alpha \in \bigwedge^p V^*$ and $\beta \in \bigwedge^q V^*$, then we define their antisymmetric tensor product

$$\alpha \wedge \beta := \tfrac{(p+q)!}{p!q!}\, \mathrm{Alt}_{p+q}(\alpha \otimes \beta) \in \bigwedge^{p+q} V^*\,. \tag{6}$$

One easily sees that

$$\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha\,. \tag{7}$$

Bilinear extension of $\wedge$ to all of $\bigwedge V^*$ endows it with the structure of a real 2^n-dimensional associative algebra, the so-called exterior algebra over V^*. If $\alpha_1, \cdots, \alpha_p$ are in V^*, we have

$$\alpha_1 \wedge \cdots \wedge \alpha_p = \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\, \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)}\,, \tag{8}$$

as one easily shows from (6) and (7) using induction.

If $\{\theta^1, \cdots, \theta^n\}$ is a basis of V^*, a basis of $\bigwedge^p V^*$ is given by the following $\binom{n}{p}$ vectors

$$\{\theta^{a_1} \wedge \cdots \wedge \theta^{a_p} \mid 1 \leq a_1 < a_2 < \cdots < a_p \leq n\}\,. \tag{9}$$

An expansion of $\alpha \in \bigwedge^p V^*$ in this basis is written as follows

$$\alpha =: \tfrac{1}{p!}\, \alpha_{a_1 \cdots a_p}\, \theta^{a_1} \wedge \cdots \wedge \theta^{a_p}\,, \tag{10}$$

using standard summation convention and where the coefficients $\alpha_{a_1 \cdots a_p}$ are totally antisymmetric in all indices. On the level of coefficients, (6) reads

$$(\alpha \wedge \beta)_{a_1 \cdots a_{p+q}} = \tfrac{(p+q)!}{p!q!} \alpha_{[a_1 \cdots a_p} \beta_{a_{p+1} \cdots a_{p+q}]}\,, \tag{11}$$

where square brackets denote total antisymmetrisation in all indices enclosed:

$$\alpha_{[a_1 \cdots a_p]} := \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\ \alpha_{a_{\sigma(1)} \cdots a_{\sigma(p)}}\,. \tag{12}$$

2 Inner products

Every non-degenerate bilinear form $\eta : V \times V \to \mathbb{R}$ on a vector space V defines an isomorphism $\eta_\downarrow : V \to V^*$ to its dual space V^* via the requirement $\eta_\downarrow(v)(w) := \eta(v, w)$ for all $v, w \in V$; in short, $v \mapsto \eta_\downarrow(v) := \eta(v, \cdot)$. Its inverse map is $\eta_\uparrow : V^* \to V$, $\eta_\uparrow := (\eta_\downarrow)^{-1}$, which in turn defines a non-degenerate bilinear form on the dual space, $\eta^{-1} : V^* \times V^* \to \mathbb{R}$, via the requirement $\eta^{-1}(\alpha, \beta) := \alpha(\eta_\uparrow(\beta))$ for all $\alpha, \beta \in V^*$. On component-level this reads as follows: Let $\{e_a \mid 1 \leq a \leq n\}$ be a basis of V and $\{\theta^a \mid 1 \leq a \leq n\}$ its dual basis of V^*, so that $\theta^a(e_b) = \delta^a_b$. Then, writing $v = v^a e_a$, we get $\eta_\downarrow(v) = v_b \theta^b$ with

$$v_b := v^a \eta_{ab} \tag{13}$$

and $\eta_{ab} := \eta(e_a, e_b)$. Similarly, writing $\alpha = \alpha_a \theta^a$, we get $\eta_\uparrow(\alpha) = \alpha^a e_a$ with

$$\alpha^a := \eta^{ab} \alpha_b \tag{14}$$

and $\eta^{ab} := \eta^{-1}(\theta^a, \theta^b)$. Note that in (13) it is the *first* index on η_{ab} that is contracted with v_a whereas in (14) it is the *second* index on η^{ab} that is contracted with α_b. This is important for consistency in case η is not symmetric.

The previous equations imply

$$\eta^{ac}\eta_{bc} = \eta^{ca}\eta_{cb} = \delta^a_b \tag{15}$$

and

$$\eta^{ab} = \eta^{ac}\eta^{bd}\eta_{cd} \tag{16a}$$

$$\eta_{ab} = \eta^{cd}\eta_{ca}\eta_{db}. \tag{16b}$$

This explains why $\eta_\uparrow$ and $\eta_\downarrow$ are called the operations of "index-raising" and "index lowering". Sometimes the images of $\eta_\uparrow$ and $\eta_\downarrow$ are indicated by the musical symbols $\sharp$ (sharp) and $\flat$ (flat) respectively, i.e., one writes $\eta_\uparrow(\alpha) = \alpha^\sharp$ and $\eta_\downarrow(v) = v^\flat$, which makes sense as long as the bilinear form η with respect to which these maps are defined is self understood. We shall also employ this notation.

We stress once more that up to this point we did not assume η to be symmetric, so that all formulae apply generally. In particular, they will apply to antisymmetric η which occur in spinor calculus. However, for the rest of these supplementary notes we will assume η to be symmetric.

The symmetric inner products on V and V^* naturally extend to symmetric inner product on tensor-product spaces, just by taking products slotwise. In particular, we have on T^pV^*

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_p, \beta_1 \otimes \cdots \otimes \beta_p \rangle := \prod_{a=1}^{p} \eta^{-1}(\alpha_a, \beta_a) \tag{17}$$

and bilinear extension:

$$\langle \alpha_{a_1\cdots a_p}\, \theta^{a_1} \otimes \cdots \otimes \theta^{a_p}\,,\, \beta_{b_1\cdots b_p}\, \theta^{b_1} \otimes \cdots \otimes \theta^{b_p} \rangle = \alpha_{a_1\cdots a_p}\beta^{a_1\cdots a_p}\,. \tag{18}$$

On each subspace $\bigwedge^p V^* \subset T^pV*$ we have

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p\,,\, \beta_1 \wedge \cdots \wedge \beta_p \rangle := p! \sum_{\sigma\in S_p} \operatorname{sign}(\sigma) \prod_{a=1}^{p} \eta(\alpha_a, \beta_{\sigma(a)}) \tag{19}$$

and hence

$$\left\langle \tfrac{1}{p!}\alpha_{a_1\cdots a_p}\theta^{a_1} \wedge \cdots \wedge \theta^{a_p}\,,\, \tfrac{1}{p!}\beta_{b_1\cdots b_p}\theta^{b_1} \wedge \cdots \wedge \theta^{b_p} \right\rangle = \alpha_{a_1\cdots a_p}\beta^{a_1\cdots a_p}\,. \tag{20}$$

In the totally antisymmetric case it is sometimes more convenient to renormalise this product in a p-dependent fashion. One sets

$$\langle \cdot\,,\,\cdot\rangle_{\mathrm{norm}}\big|_{\bigwedge^p V^*} := \tfrac{1}{p!}\,\langle \cdot\,,\,\cdot\rangle\big|_{\bigwedge^p V^*} \tag{21}$$

so that

$$\Big\langle \tfrac{1}{p!}\alpha_{a_1\cdots a_p}\theta^{a_1}\wedge\cdots\wedge\theta^{a_p}\,,\ \tfrac{1}{p!}\beta_{b_1\cdots b_p}\theta^{b_1}\wedge\cdots\wedge\theta^{b_p}\Big\rangle_{\mathrm{norm}} = \tfrac{1}{p!}\alpha_{a_1\cdots a_p}\beta^{a_1\cdots a_p}\,. \tag{22}$$

3 Hodge duality

Given a choice o of an orientation of V^* (e.g. induced by an orientation of V), there is a unique top-form $\varepsilon \in \bigwedge^n V^*$ (i.e. a *volume form* for V), associated with the triple (V^*, η^{-1}, o), given by

$$\varepsilon := \theta^1\wedge\cdots\wedge\theta^n\,, \tag{23}$$

where $\{\theta^1,\cdots,\theta^n\}$ is any η^{-1}-orthonormal Basis of V^* in the orientation class o. The *Hodge duality* map at level $0\leq p\leq n$ is a linear isomorphism

$$\star_p : \bigwedge^p V^* \to \bigwedge^{n-p} V^*\,, \tag{24a}$$

defined implicitly by

$$\alpha\wedge\star_p\beta = \varepsilon\,\langle\alpha\,,\,\beta\rangle_{\mathrm{norm}}\,. \tag{24b}$$

This means that the image of $\beta\in\bigwedge^p V^*$ under $\star_p$ in $\bigwedge^{n-p} V^*$ is defined by the requirement that (24b) holds true for all $\alpha\in\bigwedge^p V^*$. Linearity is immediate and uniqueness of $\star_p$ follows from the fact that if $\lambda\in\bigwedge^{n-p} V^*$ and $\alpha\wedge\lambda=0$ for all $\alpha\in\bigwedge^p V^*$, then $\lambda=0$. To show existence it is sufficient to define $\star_p$ on basis vectors. Since (24b) is also linear in α it is sufficient to verify (24b) if α runs through all basis vectors.

From now on we shall follow standard practice and drop the subscript p on $\star$, supposing that this will not cause confusion.

Let $\{e_1,\cdots e_n\}$ be a basis of V and $\{\theta^1,\cdots,\theta^n\}$ its dual basis of V^*; i.e. $\theta^a(e_b)=\delta^a_b$. Let further $\{\theta_1,\cdots,\theta_n\}$ be the basis of V^* given by the image of $\{e_1,\cdots e_n\}$ under $\eta_\downarrow$, i.e. $\theta_a=\eta_{ab}\theta^b$. Then, on the basis $\{\theta_{a_1}\wedge\cdots\wedge\theta_{a_p}\mid 1\leq a_1<a_2<\cdots<a_p\leq n\}$ of $\bigwedge^p V^*$ the map $\star$ has the simple form

$$\star(\theta_{b_1}\wedge\cdots\wedge\theta_{b_p}) = \tfrac{1}{(n-p)!}\varepsilon_{b_1\cdots b_p\,a_{p+1}\cdots a_n}\,\theta^{a_{p+1}}\wedge\cdots\wedge\theta^{a_n}\,. \tag{25}$$

This is proven by merely checking (24b) for $\alpha=\theta^{a_1}\wedge\cdots\wedge\theta^{a_p}$ and $\beta=\theta_{b_1}\wedge\cdots\wedge\theta_{b_p}$. Instead of (25) we can write

$$\begin{aligned}\star(\theta^{a_1}\wedge\cdots\wedge\theta^{a_p}) &= \tfrac{1}{(n-p)!}\,\eta^{a_1b_1}\cdots\eta^{a_pb_p}\,\varepsilon_{b_1\cdots b_pb_{p+1}\cdots b_n}\,\theta^{b_{p+1}}\wedge\cdots\wedge\theta^{b_n}\\ &= \tfrac{1}{(n-p)!}\,\varepsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n}\,\theta^{a_{p+1}}\wedge\cdots\wedge\theta^{a_n}\,,\end{aligned} \tag{26}$$

which makes explicit the dependence on ε and η.

If $\alpha = \frac{1}{p!}\alpha_{a_1\cdots a_p}\theta^{a_1}\wedge\cdots\wedge\theta^{a_p}$, then $\star\alpha = \frac{1}{(n-p)!}(\star\alpha)_{b_1\cdots b_{n-p}}\theta^{b_1}\wedge\cdots\wedge\theta^{b_{n-p}}$, where

$$(\star\alpha)_{b_1\cdots b_{n-p}} = \tfrac{1}{p!}\,\alpha_{a_1\cdots a_p}\varepsilon^{a_1\cdots a_p}{}_{b_1\cdots b_{n-p}}\,. \tag{27}$$

This gives the familiar expression of Hodge duality in component language. Note that on component level the first (rather than last) p indices are contracted.

Applying $\star$ twice (i.e. actually $\star_{(n-p)}\circ\star_p$) leads to the following self-map of $\bigwedge^p V^*$:

$$\begin{aligned}
&\star\big(\star(\theta^{a_1}\wedge\cdots\wedge\theta^{a_p})\big)\\
&= \tfrac{1}{p!(n-p)!}\varepsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n}\varepsilon^{a_{p+1}\cdots a_n}{}_{b_1\cdots b_p}\,\theta^{b_1}\wedge\cdots\wedge\theta^{b_p}\\
&= \tfrac{(-1)^{p(n-p)}}{p!(n-p)!}\,\varepsilon^{a_1\cdots a_p a_{p+1}\cdots a_n}\varepsilon_{b_1\cdots b_p a_{p+1}\cdots a_n}\,\theta^{b_1}\wedge\cdots\wedge\theta^{b_p}\\
&= (-1)^{p(n-p)}\,\langle\varepsilon,\varepsilon\rangle_{\text{norm}}\,\theta^{a_1}\wedge\cdots\wedge\theta^{a_p}\,.
\end{aligned} \tag{28}$$

Note that

$$\langle\varepsilon,\varepsilon\rangle_{\text{norm}} = \tfrac{1}{n!}\eta^{a_1b_1}\cdots\eta^{a_nb_n}\varepsilon_{a_1\cdots a_n}\varepsilon_{b_1\cdots b_n} = (\varepsilon_{12\cdots n})^2/\det\{\eta(e_a,e_b)\}\,. \tag{29}$$

This formula holds for any volume form ε in the definition (24b), independent of whether or not it is related to η.

Since the right-hand side of (24b) is symmetric under the exchange $\alpha\leftrightarrow\beta$, so must be the left-hand side. Using (28) we get

$$\begin{aligned}
\langle\alpha,\beta\rangle_{\text{norm}}\,\varepsilon &= \alpha\wedge\star\beta = \beta\wedge\star\alpha = (-1)^{p(n-p)}\star\alpha\wedge\beta\\
&= \langle\varepsilon,\varepsilon\rangle^{-1}_{\text{norm}}\star\alpha\wedge\star\star\beta = \langle\varepsilon,\varepsilon\rangle^{-1}_{\text{norm}}\,\langle\star\alpha\,,\,\star\beta\rangle_{\text{norm}}\,\varepsilon\,,
\end{aligned} \tag{30}$$

hence

$$\langle\star\alpha\,,\,\star\beta\rangle_{\text{norm}} = \langle\varepsilon,\varepsilon\rangle_{\text{norm}}\langle\alpha,\beta\rangle_{\text{norm}}\,. \tag{31}$$

From this and (28)) it follows for $\alpha\in\bigwedge^p V^*$ and $\beta\in\bigwedge^{n-p}V^*$, that

$$\langle\alpha,\star\beta\rangle_{\text{norm}} = \langle\varepsilon,\varepsilon\rangle^{-1}_{\text{norm}}\langle\star\alpha\,,\,\star\star\beta\rangle_{\text{norm}} = (-1)^{p(n-p)}\,\langle\star\alpha,\beta\rangle_{\text{norm}}\,. \tag{32}$$

This shows that the adjoint map of $\star$ relative to $\langle\cdot\,,\,\cdot\rangle_{\text{norm}}$ is $(-1)^{p(n-p)}\star$.

Formulae (28), (30)(31), and (32) are valid for general ε in the definition (24b). If we chose ε in the way we did, namely as the unique volume form that assigns unit volume to an oriented orthonormal frame, as does (23), then we have

$$\langle\varepsilon,\varepsilon\rangle_{\text{norm}} = (-1)^{n_-} \tag{33}$$

where n_- is the maximal dimension of subspaces in V restricted to which η is negative definite; i.e. η is of signature (n_+,n_-). Equation (31) then shows that $\star$ is an isometry for even n_- and an anti-isometry for odd n_- (as for Lorentzian η in any dimension).

Finally we note the following useful formula: If $v \in V$ let $i_v : T^p V^* \to T^{p-1} V^*$ the map which inserts v into the first tensor factor. It restricts to a map $i_v : \bigwedge^p V^* \to \bigwedge^{p-1} V^*$. Then, for any $\alpha \in \bigwedge^p V^*$, we have

$$i_v \star \alpha = \star(\alpha \wedge v^\flat)\,. \tag{34}$$

where $v^\flat := \eta_\downarrow(v)$. It suffices to prove this for basis elements $v = e_a$ of V and $\alpha = \theta^{a_1} \wedge \cdots \wedge \theta^{a_p}$ of $\bigwedge^p V^*$, which is almost immediate using (26).